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LETTER TO THE EDITOR

Reciprocal auto-Bäcklund transformations

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Abstract. Invariant Bäcklund transformations of reciprocal type are derived for a class of $(n+1)$th-order conservation laws. A new auto-Bäcklund transformation for the Harry–Dym equation is presented as a special case of the analysis.

In a recent paper (Kingston and Rogers 1982) certain reciprocal Bäcklund transformations were introduced along with associated permutability diagrams for a broad class of nonlinear evolution equations. Such transformations have subsequently been used to reduce important nonlinear boundary value problems to linear canonical form amenable to solution by integral transform methods (Rogers *et al* 1983, Rogers 1983). In a separate development, Nimmo and Crighton (1982) have classified those Bäcklund transformations of non-reciprocal type which exist for a class of second-order nonlinear parabolic equations. Here, the necessary and sufficient conditions are established for the existence of reciprocal-type auto Bäcklund transformations for a wide class of higher-order nonlinear evolution equations. The result is as follows.

Theorem. The $(n+1)$th-order conservation law

$$\partial u/\partial t+(\partial/\partial x)\{\mathscr{E}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n)\}=0 \tag{1}$$

is invariant under the Bäcklund transformation

$$\begin{aligned} &dx' = u\, dx - \mathscr{E}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n)\, dt, \qquad t'=t, \\ &u' = u^{-1}, \end{aligned} \tag{2}$$

if and only if

$$\mathscr{E}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n) = u^{1/2} G(\mathbb{D}^{(0)}(-\ln u), \mathbb{D}^{(1)}(-\ln u), \ldots, \mathbb{D}^{(n)}(-\ln u)) \tag{3}$$

where G is an odd function on $\mathbb{R}^{n+1}$ and $\mathbb{D} := -u^{-1/2}\, \partial/\partial x$.

Proof. It was shown by Kingston and Rogers (1982) that the conservation law

$$(\partial/\partial t)\{T(\partial/\partial x; \partial/\partial t; u)\} + (\partial/\partial x)\{F(\partial/\partial x; \partial/\partial t; u)\} = 0 \tag{4}$$

is transformed to the reciprocally associated conservation law

$$(\partial/\partial t')\{T'(\partial/\partial x'; \partial/\partial t'; u)\} + (\partial/\partial x')\{F'(\partial/\partial x'; \partial/\partial t'; u)\} = 0, \tag{5}$$

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by the Bäcklund transformations

$$\left.\begin{aligned} &\mathrm{d}x' = T\,\mathrm{d}x - F\,\mathrm{d}t, \qquad t' = t, \\ &T'(\partial/\partial x'; \partial/\partial t'; u) = 1/T(\mathrm{D}'; \partial'; u), \\ &F'(\partial/\partial x'; \partial/\partial t'; u) = -F(\mathrm{D}'; \partial'; u)/T(\mathrm{D}'; \partial'; u), \end{aligned}\right\} R \tag{6}$$

where

$$\mathrm{D}' := \frac{\partial}{\partial x} = \frac{1}{T'(\partial/\partial x'; \partial/\partial t'; u)} \frac{\partial}{\partial x'}, \tag{7}$$

$$\partial' := \frac{\partial}{\partial t} = \frac{F'(\partial/\partial x'; \partial/\partial t'; u)}{T'(\partial/\partial x'; \partial/\partial t'; u)} \frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \tag{8}$$

and the notation $\Phi(\partial/\partial x; \partial/\partial t; u), \Psi(\partial/\partial x'; \partial/\partial t'; u)$ indicates that $\Phi \equiv \Phi(u, u_x, u_{xx}, \ldots; u_t, u_{tt}, \ldots)$ and $\Psi \equiv \Psi(u, u_{x'}, u_{x'x'}, \ldots; u_{t'}, u_{t't'}, \ldots)$ respectively.

It is easily shown that $R^2 = I$ so that the Bäcklund transformation R is reciprocal.

The specialisation

$$T = u, \qquad F = \mathscr{E}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n) \tag{9}$$

shows that (1) is transformed under the Bäcklund transformation (2) to

$$\partial u'/\partial t' + \partial \mathscr{E}'/\partial x' = 0 \tag{10}$$

where

$$\mathscr{E}' = -u'\mathscr{E}(\mathrm{D}'^{(0)}(u'^{-1}), \mathrm{D}'^{(1)}(u'^{-1}), \ldots, \mathrm{D}'^{(n)}(u'^{-1})) \tag{11}$$

and

$$\mathrm{D}' := u'^{-1}\,\partial/\partial x'. \tag{12}$$

Hence, for invariance of (1) under the Bäcklund transformation (2) it is required that

$$\mathscr{E}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n) = -u\mathscr{E}(u^{-1}, (u^{-1}\,\partial/\partial x)u^{-1}, \ldots, (u^{-1}\,\partial/\partial x)^n u^{-1}). \tag{13}$$

In order to characterise those $\mathscr{E}$ for which this relation holds a preliminary result is needed.

Thus, we introduce functions $g_j, h_j, j = 0, 1, \ldots : \mathbb{R}^j \to \mathbb{R}$ in terms of functions $a, b, c, d: \mathbb{R} \to \mathbb{R}$, each suitably differentiable, by

$$g_j(v, \partial v/\partial y, \ldots, \partial^j v/\partial y^j) = (a(v)\partial/\partial y)^j b(v), \tag{14}$$

$$h_j(w, \partial w/\partial z, \ldots, \partial^j w/\partial z^j) = (c(w)\partial/\partial z)^j d(w), \tag{15}$$

where v and w are partially dependent on y and z respectively.

If we now set $y = \int c(w)^{-1}\,\mathrm{d}z$ and $v = d(w)$ then

$$h_j(w, \partial w/\partial z, \ldots, \partial^j w/\partial z^j) = (\partial/\partial y)^j v \tag{16}$$

so that

$$\begin{aligned} &g_j(h_0(w), h_1(w, \partial w/\partial z), \ldots, h_j(w, \partial w/\partial z, \ldots, \partial^j w/\partial z^j)) \\ &\qquad = g_j(v, \partial v/\partial y, \ldots, \partial^j v/\partial y^j) \\ &\qquad = (a(v)\,\partial/\partial y)^j b(v) \\ &\qquad = (a(d(w))c(w)\,\partial/\partial z)^j b(d(w)). \end{aligned} \tag{17}$$

Similarly,

$$g_j(-h_0(w), -h_1(w, \partial w/\partial z), \ldots, -h_j(w, \partial w/\partial z, \ldots, \partial^j w/\partial z^j))$$
$$= (a(-d(w))c(w)\, \partial/\partial z)^j b(-d(w)). \qquad (18)$$

If we now choose

$$a(v) = -\mathrm{e}^{-v/2}, \qquad b(v) = \mathrm{e}^{-v}, \qquad c(w) = -w^{-1/2}, \qquad d(w) = -\ln w \qquad (19)$$

in (17) and (18) then it is seen that

$$g_j(h_0(u), h_1(u, \partial u/\partial x), \ldots, h_j(u, \partial u/\partial x, \ldots, \partial^j u/\partial x^j)) = \partial^j u/\partial x^j \qquad (20)$$

and

$$g_j(-h_0(u), -h_1(u, \partial u/\partial x), \ldots, -h_j(u, \partial u/\partial x, \ldots, \partial^j u/\partial x^j)) = (u^{-1}\, \partial/\partial x)^j u^{-1}$$

where the functions $g_j: \mathbb{R}^{j+1} \to \mathbb{R}$ and $h_j: \mathbb{R}^{j+1} \to \mathbb{R}$ are defined by

$$g_j(v, \partial v/\partial y, \ldots, \partial^j v/\partial y^j) = (-\mathrm{e}^{-v/2}\, \partial/\partial y)^j\, \mathrm{e}^{-v} \qquad (22)$$

and

$$h_j(w, \partial w/\partial z, \ldots, \partial^j w/\partial z^j) = (-w^{-1/2}\, \partial/\partial z)^j(-\ln w), \qquad j = 0, 1, 2, \ldots. \qquad (23)$$

Thus, the invariance condition (13) which may be expressed as

$$u^{-1/2}\mathscr{E}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n)$$
$$= -u^{1/2}\mathscr{E}((u^{-1}\, \partial/\partial x)^0 u^{-1}, (u^{-1}\, \partial/\partial x)^1 u^{-1}, \ldots, (u^{-1}\, \partial/\partial x)^n u^{-1}) \qquad (24)$$

is equivalent to

$$(g_0(h_0(u)))^{-1/2}\mathscr{E}(g_0(h_0(u)), g_1(h_0(u), h_1(u, \partial u/\partial x)), \ldots,$$
$$g_n(h_0(u), \ldots, h_n(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n)))$$
$$= -(g_0(-h_0(u)))^{-1/2}\mathscr{E}(g_0(-h_0(u)), g_1(-h_0(u), -h_1(u, \partial u/\partial x)), \ldots,$$
$$g_n(-h_0(u), \ldots, -h_n(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n))). \qquad (25)$$

Hence, $u^{-1/2}\mathscr{E}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n)$ is an odd function of $h_0(u)$, $h_1(u, \partial u/\partial x), \ldots, h_n(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n)$. The relation (3) now follows from (23) and our result is established.

An auto-Bäcklund transformation of the Harry–Dym equation. If we set

$$G \equiv \mathbb{D}^2(-\ln v) = \tfrac{3}{2}u_x^2 u^{-3} - u_{xx}u^{-2} \qquad (26)$$

then insertion of (3) into (1) produces the Harry–Dym equation (Kruskal 1975)

$$u_t - 2\{u^{-1/2}\}_{xxx} = 0. \qquad (27)$$

The above shows that (27) is invariant under the reciprocal Bäcklund transformation

$$\mathrm{d}x' = u\, \mathrm{d}x + 2\{u^{-1/2}\}_{xx}\, \mathrm{d}t, \qquad t' = t,$$
$$u' = u^{-1}. \qquad (28)$$

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